Topological quantum gate entangler for a multi-qubit state

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Abstract

We establish a relation between topological and quantum entanglement for a multi-qubit state by considering the unitary representations of the Artin braid group. We construct topological operators that can entangle multi-qubit state. In particular we construct operators that create quantum entanglement for multi-qubit states based on the Segre ideal of complex multi-projective space. We also in detail discuss and construct these operators for two-qubit and three-qubit states.

1 Introduction

Multipartite entangled states are the building block of a universal quantum computer. For example an one-way quantum computer as a scheme for universal quantum computation are based on entangled cluster states. Recently, L. Kauffman and S. Lomonaco Jr. have shown that topological entanglement and quantum entanglement are closely related [1]. They introduced a topological operator called braiding operator that can entangle quantum state. These operator are solution of Yang-Baxter equation. The braiding operator are also unitary transformation which make them very suitable for application in the field of quantum computing. We have also recently establish a relation between multipartite states and the Segre variety and the Segre ideal [2, 3]. For example, we have shown that the Segre ideal represent completely separable states of multipartite states. In this paper, we will construct braiding operators for multi-qubit states based on construction of the Segre ideal. In particular, in section 2 we will give a short introduction to complex projective variety and complex multi-projective Segre variety and ideal. In section 3 we will review the basic construction of topological entanglement operators. We also discuss braiding operator for two-qubit state. Finally in section 4 we will construct such topological unitary operators for multi-qubit states. We will in detail discuss the construction of this operator for a three-qubit state. Now, denote a general, composite quantum system with m subsystems as $\mathcal{Q} = \mathcal{Q}^p_m(N_1, N_2, \dots, N_m) = \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m$, with the pure state $|\Psi\rangle = \sum_{k_1,k_2,\dots,k_m=1}^{N_1,N_2,\dots,N_m} \alpha_{k_1k_2\dots k_m}|k_1k_2\dots k_m\rangle$ and corresponding Hilbert space $\mathcal{H}_\mathcal{Q} = \mathcal{H}_{\mathcal{Q}_1}\otimes\mathcal{H}_{\mathcal{Q}_2}\otimes\cdots\otimes\mathcal{H}_{\mathcal{Q}_m}$, where the dimension of the jth Hilbert space is $N_j = \dim(\mathcal{H}_{\mathcal{Q}_j})$. We are going to use this notation throughout this paper. In particular, we denote a pure two-qubit state by $\mathcal{Q}^p_2(2,2)$. Next, let $\rho_\mathcal{Q}$ denote a density operator acting on $\mathcal{H}_\mathcal{Q}$. The density operator $\rho_\mathcal{Q}$ is said to

be fully separable, which we will denote by $\rho_{\mathcal{Q}}^{sep}$, with respect to the Hilbert space decomposition, if it can be written as $\rho_{\mathcal{Q}}^{sep} = \sum_{k=1}^{\mathrm{N}} p_k \bigotimes_{j=1}^{m} \rho_{\mathcal{Q}_j}^{k}$, $\sum_{k=1}^{\mathrm{N}} p_k = 1$ for some positive integer N, where p_k are positive real numbers and $\rho_{\mathcal{Q}_j}^{k}$ denotes a density operator on Hilbert space $\mathcal{H}_{\mathcal{Q}_j}$. If $\rho_{\mathcal{Q}}^{p}$ represents a pure state, then the quantum system is fully separable if $\rho_{\mathcal{Q}}^{p}$ can be written as $\rho_{\mathcal{Q}}^{sep} = \bigotimes_{j=1}^{m} \rho_{\mathcal{Q}_j}$, where $\rho_{\mathcal{Q}_j}$ is the density operator on $\mathcal{H}_{\mathcal{Q}_j}$. If a state is not separable, then it is said to be an entangled state.

2 Complex projective variety and Segre ideal for multi-qubit state

In this section, we will define complex projective space, ideal, and variety. Moreover, we will review the construction of the Segre ideal for multi-qubit state. Here are some general references on complex projective geometry [4, 5]. A complex projective space $\mathbb{P}_{\mathbb{C}}^{n}$ is defined to be the set of lines through the origin in $\mathbb{C}^{n+1}$, that is, $\mathbb{P}_{\mathbb{C}}^{n} = \frac{\mathbb{C}^{n+1}-0}{(x_1,\ldots,x_{n+1})\sim(y_1,\ldots,y_{n+1})}$, $\lambda \in \mathbb{C}-0$, $y_i = \lambda x_i \ \forall\ 0 \leq i \leq n+1$. Let $C[z] = C[z_1, z_2, \ldots, z_n]$ denotes the polynomial algebra in n variables with complex coefficients. Then, given a set of homogeneous polynomials $\{g_1, g_2, \ldots, g_q\}$ with $g_i \in C[z]$, we define a complex projective variety as

$$\mathcal{V}(g_1, \ldots, g_q) = \{O \in \mathbb{P}_{\mathbb{C}}^{n} : g_i(O) = 0 \ \forall\ 1 \leq i \leq q\}, \tag{2.0.1}$$

where $O = [a_1, a_2, \ldots, a_{n+1}]$ denotes the equivalent class of point $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\} \in \mathbb{C}^{n+1}$. Let $\mathcal{V}$ be complex projective variety. Then an ideal of $\mathbf{C}[z_1, z_2, \ldots, z_n]$ is defined by

$$\mathcal{I}(\mathcal{V}) = \{g \in \mathbf{C}[z_1, z_2, \ldots, z_n] : g(z) = 0 \ \forall\ z \in \mathcal{V}\}. \tag{2.0.2}$$

Note also that $\mathcal{V}(\mathcal{I}(\mathcal{V})) = \mathcal{V}$. We can map the product of spaces $\mathbb{P}_{\mathbb{C}}^{N_1-1} \times \mathbb{P}_{\mathbb{C}}^{N_2-1} \times \cdots \times \mathbb{P}_{\mathbb{C}}^{N_m-1}$ into a projective space by its Segre embedding as follows. Let $(\alpha_1^i, \alpha_2^i, \ldots, \alpha_{N_i}^i)$ be points defined on the complex projective space $\mathbb{P}_{\mathbb{C}}^{N_i-1}$. Then the Segre map

$$\mathbb{P}_{\mathbb{C}}^{N_1-1} \times \mathbb{P}_{\mathbb{C}}^{N_2-1} \times \cdots \times \mathbb{P}_{\mathbb{C}}^{N_m-1} \xrightarrow[\mathcal{S}_{N_1,\ldots,N_m}]{} \mathbb{P}_{\mathbb{C}}^{N_1 N_2 \cdots N_m - 1} \tag{2.0.3}$$

is defined by $((\alpha_1^1, \alpha_2^1, \ldots, \alpha_{N_1}^1), \ldots, (\alpha_1^m, \alpha_2^m, \ldots, \alpha_{N_m}^m)) \longmapsto (\alpha_{i_1}^1 \alpha_{i_2}^2 \cdots \alpha_{i_m}^m)$. Next, let $\alpha_{i_1 i_2 \cdots i_m}$,$1 \leq i_j \leq N_j$ be a homogeneous coordinate-function on $\mathbb{P}_{\mathbb{C}}^{N_1 N_2 \cdots N_m - 1}$. For a multi-qubit quantum system the Segre ideal is defined by

$$\mathcal{I}_{\mathrm{Segre}}^{m} = \sum_{j=1}^{m} \mathcal{I}_{\mathcal{Q}_j \models \mathcal{Q}_1 \mathcal{Q}_2 \cdots \widehat{\mathcal{Q}}_j \cdots \mathcal{Q}_m}, \tag{2.0.4}$$

where $\mathcal{I}_{\mathcal{Q}_j \models \mathcal{Q}_1 \mathcal{Q}_2 \cdots \widehat{\mathcal{Q}}_j \cdots \mathcal{Q}_m}$ is the ideal defining when a subsystem $\mathcal{Q}_j$ is separated from quantum system $\mathcal{Q}_1 \mathcal{Q}_2 \cdots \mathcal{Q}_m$ is generated by

$$\mathcal{I}_{\mathcal{Q}_j \models \mathcal{Q}_1 \mathcal{Q}_2 \cdots \widehat{\mathcal{Q}}_j \cdots \mathcal{Q}_m} = \left\langle \mathrm{Minors}_{2\times 2} \mathcal{X}_{2\times 2^{m-1}}^{j} \right\rangle, \tag{2.0.5}$$

where $\mathcal{X}^{j}_{2\times 2^{m-1}}$ is the following $2\times 2^{m-1}$ matrix

$$\begin{pmatrix} \alpha_{11\ldots11_j1\ldots1} & \alpha_{11\ldots11_j1\ldots2} & \cdots & \alpha_{22\ldots21_j2\ldots2} \\ \alpha_{11\ldots12_j1\ldots1} & \alpha_{11\ldots12_j1\ldots2} & \cdots & \alpha_{22\ldots22_j2\ldots2} \end{pmatrix}. \tag{2.0.6}$$

where $j=1,2,\ldots,m$ and $\mathcal{Q}_j \models \mathcal{Q}_1\mathcal{Q}_2\cdots\widehat{\mathcal{Q}}_j\cdots\mathcal{Q}_m$ means we delete $\mathcal{Q}_j$ from right side and add it to the left side of $\models$.

3 Topological entanglement operators

In this section we will give a short introduction to Artin braid group and Yang-Baxter equation. We will study relation between topological and quantum entanglement by investigating the unitary representation of Artin braid group. Here are some general references on quantum group and low-dimensional topology [6, 7]. The Artin braid group B_n on n strands is generated by $\{b_n : 1 \leq i \leq n-1\}$ and we have the following relations in the group B_n: i) $b_ib_j = b_jb_i$ for $|i-j| \geq n$ and ii) $b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1}$ for $1 \leq i < n$. Let $\mathcal{V}$ be a complex vector space. Then, for two strand braid there is associated an operator $\mathcal{R}: \mathcal{V}\otimes\mathcal{V} \longrightarrow \mathcal{V}\otimes\mathcal{V}$. Moreover, let $\mathcal{I}$ be the identity operator on $\mathcal{V}$. Then, the Yang-Baxter equation is defined by

$$(\mathcal{R}\otimes\mathcal{I})(\mathcal{I}\otimes\mathcal{R})(\mathcal{R}\otimes\mathcal{I}) = (\mathcal{I}\otimes\mathcal{R})(\mathcal{R}\otimes\mathcal{I})(\mathcal{I}\otimes\mathcal{R}). \tag{3.0.7}$$

The Yang-Baxter equation represents the fundamental topological relation in the Artin braid group. The inverse to $\mathcal{R}$ will be associated with the reverse elementary braid on two strands. Next, we define a representation τ of the Artin braid group to the automorphism of $\mathcal{V}^{\otimes m} = \mathcal{V}\otimes\mathcal{V}\otimes\cdots\otimes\mathcal{V}$ by

$$\tau(b_i) = \mathcal{I}\otimes\cdots\otimes\mathcal{I}\otimes\mathcal{R}\otimes\mathcal{I}\otimes\cdots\otimes\mathcal{I}, \tag{3.0.8}$$

where $\mathcal{R}$ are in position i and $i+1$. This equation describe a representation of the braid group if $\mathcal{R}$ satisfies the Yang-Baxter equation and is also invertible. Moreover, this representation of braid group is unitary if $\mathcal{R}$ is also unitary operator. Thus $\mathcal{R}$ being unitary indicated that this operator can performs topological entanglement and it also can be considers as quantum gate. It has been show in [1] that $\mathcal{R}$ can also perform quantum entanglement by acting on qubits states. Now, let $\alpha_{11}, \alpha_{12}, \alpha_{21}$, and α_{22} be any scalars on the unit circle in the complex plane. Then, we can construct an unitary $\mathcal{R}$ as follow

$$\mathcal{R} = \begin{pmatrix} \alpha_{11} & 0 & 0 & 0 \\ 0 & 0 & \alpha_{12} & 0 \\ 0 & \alpha_{21} & 0 & 0 \\ 0 & 0 & 0 & \alpha_{22} \end{pmatrix} \tag{3.0.9}$$

which is a solution to the Yang-Baxter equation. To see how it is related to quantum gates, let $\mathcal{P}$ be the swap gate $\tau = \mathcal{R}\mathcal{P}$ gate define by

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} \alpha_{11} & 0 & 0 & 0 \\ 0 & \alpha_{12} & & 0 \\ 0 & 0 & \alpha_{21} & 0 \\ 0 & 0 & 0 & \alpha_{22} \end{pmatrix} \tag{3.0.10}$$

In view of braiding and algebra, $\mathcal{R}$ is a solution to the braided version of the Yang-Baxter equation and τ is a solution to the algebraic Yang-Baxter equation and $\mathcal{P}$ represent a virtual or flat crossing. The action of unitary matrix $\mathcal{R}$ on a quantum state are: i) $\mathcal{R}|11\rangle = \alpha_{11}|11\rangle$, ii) $\mathcal{R}|12\rangle = \alpha_{21}|21\rangle$, iii) $\mathcal{R}|21\rangle = \alpha_{12}|12\rangle$, iv) $\mathcal{R}|22\rangle = \alpha_{22}|22\rangle$. A proof that the operator $\mathcal{R}$ can entangle quantum states is give in [1]. Here, we will also give a proof based on the construction of the Segre variety.

Lemma 3.0.1 *If elements of $\mathcal{R}$ satisfies $\alpha_{11}\alpha_{22} \neq \alpha_{12}\alpha_{21}$, then the state $\mathcal{R}(|\psi\rangle \otimes |\psi\rangle)$, with $|\psi\rangle = |1\rangle + |2\rangle$ is entangled.*

From the construction of the Segre ideal the separable set of two qubit state satisfies $\alpha_{11}\alpha_{22} = \alpha_{12}\alpha_{21}$. Thus a two qubit state

$$\mathcal{R}(|\psi\rangle \otimes |\psi\rangle) = \alpha_{11}|11\rangle + \alpha_{12}|12\rangle + \alpha_{21}|21\rangle + \alpha_{22}|22\rangle. \tag{3.0.11}$$

is entangled if and only if this inequality does not hold. We can also note that a measure of entanglement for two-qubit state in give by concurrence $\mathcal{C}(|\Phi\rangle) = 2|\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}|$. In general, let $M = (\mathcal{M}_{kl})$ denote an $n \times n$ matrix with complex elements and let $\mathcal{R}$ be defined by $\mathcal{R}^{kl}_{rs} = \delta^k_s \delta^l_r \mathcal{M}_{kl}$. Then $\mathcal{R}$ is a unitary solution to the Yang-Baxter equation. In the next section, we will used this construction and proof to create entangled states for three-qubit states.

4 Multi-qubit quantum gate entangler

In the previous section we have shown that how we can create entangled state using topological unitary transformation $\mathcal{R}$. We have also show a relation between the Segre ideal and such transformation. In this section, we will use this information to construct multi-qubit entangled state based on this ideal. But first, we will construct such topological operator for three-qubit state. For this state the ideal $\mathcal{I}^{2,2,2}_{\mathcal{Q}_1 \models \mathcal{Q}_2\mathcal{Q}_3}$ representing if a subsystem $\mathcal{Q}_1$ that is unentangled with $\mathcal{Q}_2\mathcal{Q}_3$ is generated by $\mathcal{I}_{\mathcal{Q}_1 \models \mathcal{Q}_2\mathcal{Q}_3} = \left\langle \mathrm{Minors}_{2\times 2}\mathcal{X}^1_{2\times 4}\right\rangle$, that is

$$\mathcal{I}_{\mathcal{Q}_1 \models \mathcal{Q}_2\mathcal{Q}_3} = \left\langle \mathrm{Minors}_{2\times 2}\begin{pmatrix} \alpha_{111} & \alpha_{112} & \alpha_{121} & \alpha_{122} \\ \alpha_{211} & \alpha_{212} & \alpha_{221} & \alpha_{222} \end{pmatrix}\right\rangle, \tag{4.0.12}$$

where we have used the notation $\models$ to indicate that $\mathcal{Q}_1$ is separated from $\mathcal{Q}_2\mathcal{Q}_3$ but we still could have entanglement between $\mathcal{Q}_2$ and $\mathcal{Q}_3$. In the same way, we can define the ideal $\mathcal{I}^{2,2,2}_{\mathcal{Q}_2 \models \mathcal{Q}_1\mathcal{Q}_3}$ representing if the subsystem $\mathcal{Q}_2$ is unentangled with $\mathcal{Q}_1\mathcal{Q}_3$ and $\mathcal{I}_{\mathcal{Q}_3 \models \mathcal{Q}_1\mathcal{Q}_2}$ representing if the subsystem $\mathcal{Q}_3$ is unentangled with $\mathcal{Q}_2\mathcal{Q}_3$. The ideals are generated $\mathcal{I}_{\mathcal{Q}_2 \models \mathcal{Q}_1\mathcal{Q}_3} = \left\langle \mathrm{Minors}_{2\times 2}\mathcal{X}^2_{2\times 4}\right\rangle$, and $\mathcal{I}_{\mathcal{Q}_3 \models \mathcal{Q}_1\mathcal{Q}_2} = \left\langle \mathrm{Minors}_{2\times 2}\mathcal{X}^3_{2\times 4}\right\rangle$. Thus, the Segre ideal for three-qubit state is given by

$$\begin{aligned} \mathcal{I}^3_{\mathrm{Segre}} &= \mathcal{I}_{\mathcal{Q}_1 \models \mathcal{Q}_2\mathcal{Q}_3} + \mathcal{I}_{\mathcal{Q}_2 \models \mathcal{Q}_1\mathcal{Q}_3} + \mathcal{I}_{\mathcal{Q}_3 \models \mathcal{Q}_1\mathcal{Q}_2} \\ &= \langle \mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_{12}\rangle, \end{aligned} \tag{4.0.13}$$

where $\mathrm{T}_1 = \alpha_{1,1,1}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,1,1}$, $\mathrm{T}_2 = \alpha_{1,1,2}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,1,2}$, $\mathrm{T}_3 = \alpha_{1,1,1}\alpha_{2,1,2} - \alpha_{1,1,2}\alpha_{2,1,1}$, $\mathrm{T}_4 = \alpha_{1,2,1}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,2,1}$, $\mathrm{T}_5 = \alpha_{1,1,1}\alpha_{1,2,2} - \alpha_{1,1,2}\alpha_{1,2,1}$, $\mathrm{T}_6 = \alpha_{2,1,1}\alpha_{2,2,2} - \alpha_{2,1,2}\alpha_{2,2,1}$, $\mathrm{T}_7 = \alpha_{1,1,1}\alpha_{2,2,2} - \alpha_{1,1,2}\alpha_{2,2,1}$, $\mathrm{T}_8 = \alpha_{1,1,1}\alpha_{2,2,2} - \alpha_{1,2,1}\alpha_{2,1,2}$, $\mathrm{T}_9 = \alpha_{1,1,1}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,1,1}$, $\mathrm{T}_{10} = \alpha_{1,1,2}\alpha_{2,2,1} -$

$\alpha_{1,2,1}\alpha_{2,1,2}$, $\mathrm{T}_{11} = \alpha_{1,2,1}\alpha_{2,1,2} - \alpha_{1,2,2}\alpha_{2,1,1}$, and $\mathrm{T}_{12} = \alpha_{1,2,1}\alpha_{2,1,2} - \alpha_{1,2,2}\alpha_{2,1,1}$. In our recent paper [2] we have shown that we can construct a measure of entanglement for three-qubit states based on these Segre varieties. We have also construct a measure of entanglement for general multipartite states based on an extension of the Segre varieties [3]. For example, for three-qubit state a measure of entanglement is given by

$$\begin{aligned} \mathcal{C}(|\Psi\rangle) &= (2(|\mathrm{T}_1|^2 + |\mathrm{T}_2|^2 + |\mathrm{T}_3|^2 + |\mathrm{T}_4|^2 + |\mathrm{T}_5|^2 + |\mathrm{T}_6|^2) \\ &\quad + |\mathrm{T}_7|^2 + |\mathrm{T}_8|^2 + |\mathrm{T}_9|^2 + |\mathrm{T}_{10}|^2 + |\mathrm{T}_{11}|^2 + |\mathrm{T}_{12}|^2)^{\frac{1}{2}}, \end{aligned}$$

Now, based on comparison with the two-qubit case we will construct a unitary transformation $\mathcal{R}$ that create three-qubit entangled states. Let

$$\mathcal{R} = \begin{pmatrix} \alpha_{111} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{221} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_{212} & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_{211} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{122} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{121} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{112} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{222} \end{pmatrix}. \tag{4.0.14}$$

Then we have the following lemma for three-qubit states.

Lemma 4.0.2 *If elements of $\mathcal{R}$ satisfies $\mathrm{T}_i \neq 0$, for $1 \leq i \leq 12$, then the state $\mathcal{R}(|\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle)$, with $|\psi\rangle = |1\rangle + |2\rangle$ is entangled.*

The proof of this lemma follows by construction of $\mathcal{R}$ which is based on separable elements of three-qubit states defined by T_i. For example $\mathcal{R}(|\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle) = \sum_{k_1,k_2,k_3=1}^{2} \alpha_{k_1k_2k_3}|k_1k_2k_3\rangle$ is entangled if and only if $T_i \neq 0$. Note that we can also write the braiding operator $\mathcal{R}$ for three-qubit as $\mathcal{R}_{2^3\times 2^3} = \mathcal{R}^{d}_{2^3\times 2^3} + \mathcal{R}^{ad}_{2^3\times 2^3}$, where $\mathcal{R}^{a}_{2^3\times 2^3} = (\alpha_{111}, 0, \ldots, 0, \alpha_{222})$ is a diagonal matrix and $\mathcal{R}^{ad}_{2^3\times 2^3} = (0, \alpha_{221}, \alpha_{212}, \ldots, \alpha_{112}, 0)$ is an anti-diagonal matrix. We will use this notation to construct the matrix $\mathcal{R}$ for multi-qubit state. For m-qubit state a topological unitary transformation $\mathcal{R}_{2^m\times 2^m}$ that create multi-qubit entangled states is defined by $\mathcal{R}_{2^m\times 2^m} = \mathcal{R}^{d}_{2^m\times 2^m} + \mathcal{R}^{ad}_{2^m\times 2^m}$, where $\mathcal{R}^{a}_{2^m\times 2^m} = (\alpha_{1\cdots 1}, 0, \ldots, 0, \alpha_{2\cdots 2})$ is a diagonal matrix and $\mathcal{R}^{ad}_{2^m\times 2^m} = (0, \alpha_{22\cdots 1}, \ldots, \alpha_{21\cdots 1}, \alpha_{12\cdots 2}, \ldots, \alpha_{1\cdots 12}, 0)$ is an anti-diagonal matrix. Then we have following lemma for general multi-qubit states.

Lemma 4.0.3 *Let $\mathcal{X}^{j}_{2\times 2^{m-1}}$ be a $2 \times 2^{m-1}$ matrix defined by*

$$\mathcal{X}^{j}_{2\times 2^{m-1}} = \begin{pmatrix} \alpha_{11\ldots 11_j1\ldots 1} & \alpha_{11\ldots 11_j1\ldots 2} & \cdots & \alpha_{22\ldots 21_j2\ldots 2} \\ \alpha_{11\ldots 12_j1\ldots 1} & \alpha_{11\ldots 12_j1\ldots 2} & \cdots & \alpha_{22\ldots 22_j2\ldots 2} \end{pmatrix}. \tag{4.0.15}$$

Then the state $\mathcal{R}_{2^m\times 2^m}(|\psi\rangle_1 \otimes |\psi\rangle_2 \otimes \cdots \otimes |\psi\rangle_m)$, with $|\psi\rangle_j = |1\rangle + |2\rangle$ is entangled if 2×2 minors of $\mathcal{X}^{j}_{2\times 2^{m-1}}$ not vanishes, for all $j = 1, 2, \ldots, m$.

The proof follows by construction of $\mathcal{R}_{2^m\times 2^m}$ which is based on completely separable elements of multi-qubit states defined by the Segre ideal $\mathcal{I}^{m}_{\text{Segre}} = \sum_{j=1}^{m} \mathcal{I}_{\mathcal{Q}_j \models \mathcal{Q}_1\mathcal{Q}_2\cdots\widehat{\mathcal{Q}}_j\cdots\mathcal{Q}_m}$. That is the state $\mathcal{R}_{2^m\times 2^m}(|\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle) =$

$\sum_{k_1,k_2,\ldots,k_m=1}^{2,2,\ldots,2} \alpha_{k_1 k_2 \ldots k_m} |k_1 k_2 \ldots k_m\rangle$ is entangled if and only if all 2×2 minors of $\mathcal{X}^{j}_{2\times 2^{m-1}} \neq 0$. Note that this operator is a quantum gate entangler since $\tau_{2^m \times 2^m} = \mathcal{R}_{2^m \times 2^m} \mathcal{P}_{2^m \times 2^m}$ is a $2^m \times 2^m$ phase gate and $\mathcal{P}_{2^m \times 2^m}$ is $2^m \times 2^m$ swap gate. Thus we have succeeded to construct quantum gate entagler for a general multi-qubit state based on a similar construct of a braiding operator that satisfies condition for separability that is given by definition of the Segre ideal. This also shows a good relation between topology, algebraic geometry and quantum theory with application in the field of quantum computing.

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